

ON THE EXACT ASYMPTOTICS OF EXIT TIME FROM A CONE OF AN ISOTROPIC α -SELF-SIMILAR MARKOV PROCESS WITH A SKEW-PRODUCT STRUCTURE

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ABSTRACT. In this paper we identify the asymptotic tail of the distribution of the exit time τ_C from a cone C of an isotropic α -self-similar Markov process X_t with a skew-product structure, that is X_t is a product of its radial process and independent time changed angular component Θ_t . Under some additional regularity assumptions, the angular process Θ_t killed on exiting from the cone C has the transition density that could be expressed in terms of a complete set of orthogonal eigenfunctions with corresponding eigenvalues of an appropriate generator. Using this fact and some asymptotic properties of the exponential functional of a killed Lévy process related with Lamperti representation of the radial process, we prove that

$$\mathbb{P}_x(\tau_C > t) \sim h(x)t^{-\kappa_1}$$

as $t \rightarrow \infty$ for h and κ_1 identified explicitly. The result extends the work of DeBlassie [6] and Bañuelos and Smits [1] concerning the Brownian motion.

KEYWORDS. α -self-similar process \star cone \star exit time \star skew-product structure \star Lamperti representation \star exponential functional \star Brownian motion.

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1. INTRODUCTION

For a dimension $d \geq 2$ and an index $\alpha > 0$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ we consider an \mathbb{R}^d -valued **α -self-similar isotropic Markov process** $\{X_t, t \geq 0\}$, where $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | X_0 = x)$. We recall that process X is said to be α -self-similar if for every $x \in \mathbb{R}^d$ and $\lambda > 0$,

the law of $(\lambda X_{\lambda^{-\alpha}t}, t \geq 0)$ under \mathbb{P}_x is the same as $\mathbb{P}_{\lambda x}$.

Moreover, this process is said to be isotropic (or $O(d)$ -invariant), if for any $x \in \mathbb{R}^d$ and $\varrho \in O(d)$,

the law of $(\varrho(X_t), t \geq 0)$ under \mathbb{P}_x is the same as $\mathbb{P}_{\varrho(x)}$,

where $O(d)$ is the group of orthogonal transformations on \mathbb{R}^d . In this paper we assume that the radial process $R_t = |X_t|$ and the angular process X_t/R_t do not jump at the same time. Then by Liao and Wang [16, Theorem 1] the process X_t observed up to its first hitting time of 0 has a **skew-product structure**:

$$(1) \quad X_t = R_t \Theta_{A(t)},$$

where $A(t)$ is a strictly increasing continuous process defined by

$$(2) \quad A(t) = \int_0^t R_s^{-\alpha} ds$$

and Θ_t is an $O(d)$ -invariant Markov process on the unit sphere S^{d-1} and is **independent** of the radial process R_t . The classical example concerns d -dimensional Brownian motion that may be expressed as a product of a Bessel process and a time changed spherical Brownian motion. Moreover, the Bessel process is independent of the spherical Brownian motion. More generally, any continuous isotropic Markov process will have above representation (1) with possibly different time change; see [9]. In particular, a self-similar diffusion will have it. Note that an isotropic self-similar Markov process might not satisfy above representation (1) though. The most famous examples are the symmetric $(1/\alpha)$ -stable Lévy processes for $\alpha > 1/2$. Their Lévy measures are absolutely continuous on $\mathbb{R}^d \setminus \{0\}$, so their radial and angular parts may jump together, and thus do not possess a skew product structure as defined above.

We will also consider an **open cone** C in \mathbb{R}^d generated by a domain D in the unit sphere S^{d-1} , that is $C = \cup_{r>0} rD$. We define the **first exit time** of X_t from the cone C by

$$(3) \quad \tau_C = \inf\{t > 0 : X_t \notin C\}.$$

The purpose of this paper is to study the **asymptotic behavior of the exit probability** $\mathbb{P}_x(\tau_C > t)$ as $t \rightarrow \infty$ for $x \in C$. In fact we prove that

$$(4) \quad \mathbb{P}_x(\tau_C > t) \sim h(x)t^{-\kappa_1}$$

as $t \rightarrow \infty$ for h and κ_1 identified explicitly, where we write $f(t) \sim g(t)$ for some positive functions f and g iff $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$.

The main idea of the proof is based on the following steps. In the first one we give the following representation

$$q_D(t, \theta, \eta) = \sum_{j=1}^{\infty} e^{-\lambda_j t} m_j(\theta) m_j(\eta),$$

of the transition density for the angular process Θ_t killed upon exiting from the cone C in terms of orthogonal eigenfunctions m_j with corresponding eigenvalues λ_j of $-S|_D$ for the generator \mathcal{S} of Θ_t restricted to D with Dirichlet boundary condition. Then

$$\mathbb{P}_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|) \left(\int_D m_j d\sigma \right) \mathbb{E}_{|x|} \left[e^{-\lambda_j A(t)}, t < T_0 \right]$$

for σ being the normalized surface measure on S^{d-1} , where

$$(5) \quad T_0 = \inf\{t > 0 : X_t = 0\} = \inf\{t > 0 : R_t = 0\}.$$

Using Lamperti [12] transformation we can express process $\{R_t, t < T_0\}$ as a time change of the exponential of an $\mathbb{R} \cup \{-\infty\}$ -valued Lévy process, that is, there exists an $\mathbb{R} \cup \{-\infty\}$ -valued Lévy process ξ_t starting from 0 and with lifetime ζ , whose law does not depend on $|x|$, such that

$$(6) \quad R_t = |x| \exp(\xi_{A(t)}), \quad 0 \leq t < T_0.$$

This gives the following representation of the tail exit probability:

$$\mathbb{P}_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|) \left(\int_D m_j d\sigma \right) \mathbb{P}\left(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t\right),$$

where

$$(7) \quad I_t(\alpha\xi) = \int_0^t \exp(\alpha\xi_s) ds$$

and $I(\alpha\xi) = \lim_{t \rightarrow \zeta} I_t(\alpha\xi)$ is an exponential functional. The final result (4) follows from Rivero [19, Lemma 4] and Maulik and Zwart [18, Theorem 3.1] concerning the tail asymptotics of the exponential functional $I(\alpha\xi)$. In this case κ_1 solves equation

$$(8) \quad \phi(\alpha\kappa_1) = \lambda_1$$

for the Laplace exponent of the process ξ .

Our main result (4) extends the work of DeBlassie [6] and Bañuelos and Smits [1] concerning the Brownian motion (see also [11] for α -stable process case).

The asymptotics (4) determines also the critical exponents of integrability of the exit time τ_C . In this sense it generalizes series of papers concerning α -stable process, see Kulczycki [13] and Bañuelos and Bogdan [2] and references therein.

The paper is organized as follows. In Preliminaries we give and prove main facts used later. In the next Section 3 we give the main result and its proof.

2. PRELIMINARIES

2.1. Skew-product structure. Let X_t be an α -self-similar isotropic Markov process with skew-product representation (1). The process Θ_t is an $O(d)$ -invariant Markov process on S^{d-1} with transition semigroup Q_t and infinitesimal generator \mathcal{S} .

Throughout this paper, we assume that

Assumption 1. Θ_t possesses a bounded transition density $q(t, \theta, \eta)$ with respect to σ , the normalized surface measure on S^{d-1} , and there exist positive constants C and β such that

$$(9) \quad q(t, \theta, \eta) \leq Ct^{-\beta}$$

for all $(t, \theta, \eta) \in (0, \infty) \times S^{d-1} \times S^{d-1}$.

Example 2.1 (Brownian motion). In the case when Θ_t is a Brownian motion the Assumption 1 is satisfied. Indeed, the generator \mathcal{S} of Θ_t is a multiple of the Laplace-Beltrami operator $\Delta_{S^{d-1}}$ on S^{d-1} . Moreover, it is known that the transition density $h(t, \theta, \eta)$ of Θ_t has the Gaussian upper bound:

$$(10) \quad h(t, \theta, \eta) \leq c_1 t^{-\frac{d-1}{2}} e^{-c_2 \frac{d(\theta, \eta)^2}{t}}, \quad t > 0, \theta, \eta \in S^{d-1}$$

for some positive constants c_1 and c_2 .

Example 2.2 (Subordinate Brownian motion). Fix $\gamma \in (0, 1)$. Let W_t be a Brownian motion on S^{d-1} with transition density $h(t, \theta, \eta)$. Let S_t be a γ -stable subordinator, i.e., a Lévy process in \mathbb{R} , supported by $[0, \infty)$, with Laplace transform

$$\mathbb{E}[e^{-\vartheta S_t}] = \exp(-t\vartheta^\gamma), \quad \vartheta > 0.$$

The Assumption 1 is also satisfied when $\Theta_t = W_{S_t}$. Indeed, in this case Θ_t is an $O(d)$ -invariant pure jump Markov process on S^{d-1} with transition density

$$q(t, \theta, \eta) = \int_0^\infty h(u, \theta, \eta) p_t(u) du,$$

where $p_t(u)$ is the probability density of S_t . By Theorem 37.1 of Doetsch [8],

$$(11) \quad \lim_{u \rightarrow \infty} p_1(u) u^{1+\gamma} = \frac{\gamma}{\Gamma(1-\gamma)}.$$

The limit (11) together with the scaling property:

$$p_t(u) = t^{-\frac{1}{\gamma}} p_1(t^{-\frac{1}{\gamma}} u)$$

give the following upper bound:

$$p_t(u) \leq c_1 t u^{-1-\gamma}, \quad t, u > 0.$$

Using (10) one can observe now that

$$q(t, \theta, \eta) \leq c_3 t^{-\frac{d-1}{2\gamma}} \wedge \frac{t}{d(\theta, \eta)^{d-1+2\gamma}}$$

for $(t, \theta, \eta) \in (0, \infty) \times S^{d-1} \times S^{d-1}$.

We will now give sufficient conditions for the Assumption 1 to be satisfied for general $O(d)$ -invariant Markov process Θ_t on S^{d-1} . If we identify S^{d-1} with $O(d)/O(d-1)$ then Θ_t may be viewed as a Lévy process on the compact homogeneous space $O(d)/O(d-1)$. Furthermore, the generator \mathcal{S} of Θ_t was given by Hunt [10] (see also Liao [14]) explicitly. We state this result as follows. Let $C^\infty(S^{d-1})$ be the space of smooth functions on S^{d-1} and let $\pi : O(d) \rightarrow S^{d-1}$ be the map $g \rightarrow g\bar{o}$ for $\bar{o} = (0, \dots, 0, 1) \in \mathbb{R}^d$. Restricted to a sufficient small neighborhood V of \bar{o} , the map

$$\varphi : (y_1, \dots, y_d) \rightarrow \pi(e^{\sum_{j=1}^d y_j O_j})$$

is a diffeomorphism and (y_1, \dots, y_d) may be used as local coordinates on $\varphi(V)$, where (O_1, \dots, O_d) is a basis of Lie algebra of $O(d)$. Then by [14, Theorem 2.2] the domain of \mathcal{S} contains $C^\infty(S^{d-1})$ and for $f \in C^\infty(S^{d-1})$,

$$(12) \quad \mathcal{S}f(o) = Tf(o) + \int_{S^{d-1}} \left(f(\theta) - f(o) - \sum_{j=1}^d y_j(\theta) \frac{\partial f(o)}{\partial y_j} \right) \nu(d\theta),$$

where o is the origin on S^{d-1} , T is an $O(d)$ -invariant second order differential operator on S^{d-1} and ν is an $O(d-1)$ -invariant measure on S^{d-1} , called the Lévy measure of Θ_t , that satisfies $\nu(\{o\}) = 0$ and

$$(13) \quad \int_{S^{d-1}} [\text{dist}(\theta, o)]^2 \nu(d\theta) < \infty.$$

Since $O(d)/O(d-1)$ is irreducible, all the $O(d)$ -invariant second order differential operators are multiples of $\Delta_{S^{d-1}}$. Therefore we may rewrite (12) as

$$(14) \quad \mathcal{S}f(o) = a \Delta_{S^{d-1}} f(o) + \int_{S^{d-1}} \left(f(\theta) - f(o) - \sum_{j=1}^d y_j(\theta) \frac{\partial f(o)}{\partial y_j} \right) \nu(d\theta)$$

for some $a \geq 0$.

Note that when Θ_t is a subordinate Brownian motion defined in Example 2.2, we have $a = 0$ and the Lévy measure

$$\nu \asymp d(\theta, o)^{-d+1-2\gamma} \quad \text{near } o.$$

As observed in this case the Assumption 1 holds true.

This phenomenon holds for more general Θ_t . Using [15, Theorems 3 and 6] we can state the following proposition giving sufficient conditions for the Assumption 1 to hold true.

Proposition 2.3. *Θ_t is a Lévy process on S^{d-1} with the infinitesimal generator \mathcal{S} given by (14). Assume that either $a > 0$ or the Lévy measure ν is asymptotically larger than $d(\theta, o)^{-\gamma}$ near $\theta = o$ for some $\gamma \in (d-1, d-1+2)$. Then Θ_t has a bounded transition density $q(t, \theta, \eta)$ and it satisfies Assumption 1.*

For any open subset $D \subset S^{d-1}$ we define the first exit time of Θ_t from D by

$$(15) \quad \tau_D^\Theta = \inf\{t > 0 : \Theta_t \notin D\}.$$

Let Θ^D be the killed process of Θ upon exiting from D , that is, $\Theta_t^D = \Theta_t$ if $t < \tau_D^\Theta$ and $\Theta_t^D = \partial$ if $t \geq \tau_D^\Theta$, where ∂ is a cemetery state. Its infinitesimal generator is $\mathcal{S}|_D$, the restriction of \mathcal{S} to D with the Dirichlet boundary condition. Then

$$(16) \quad q_D(t, \theta, \eta) = q(t, \theta, \eta) - \mathbb{E}_\theta \left[q(t - \tau_D^\Theta, \Theta_{\tau_D^\Theta}, \eta); \tau_D^\Theta < t \right]$$

is the transition density of Θ^D . Clearly, $q_D(t, \theta, \eta) \leq q(t, \theta, \eta)$ for all $t > 0$ and $\theta, \eta \in S^{d-1}$. As a consequence, the transition semigroup Q_t^D associated to the subprocess Θ^D is compact on L^2 . Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues of $-\mathcal{S}|_D$ written in increasing order and repeated according to its multiplicity, and m_j the corresponding eigenfunctions normalized by $\|m_j\|_2 = 1$. Then by [5, Theorem 2.1.4], $m_j \in L^\infty$ for all j and Q_t^D has a transition density $q_D(t, \theta, \eta)$, which can be represented as the series:

$$(17) \quad q_D(t, \theta, \eta) = \sum_{j=1}^{\infty} e^{-\lambda_j t} m_j(\theta) m_j(\eta)$$

that converges uniformly on $[\delta, \infty) \times D \times D$ for all $\delta > 0$.

Remark 2.4. Note that we do not assume any regularity condition on the boundary ∂D of D . Thus $q_D(t, \theta, \eta)$ (or $m_j(\theta)$) need not vanish continuously on the boundary ∂D .

Remark 2.5. If $S^{d-1} \setminus \overline{D}$ is not empty, then from the monotonicity of Dirichlet eigenvalues we have that $\lambda_1 > 0$; see [3, Section I.5] for more details (check also Lemma 2.7 below).

Remark 2.6. Assume that $q_D(t, \theta, \eta)$ is strictly positive for $t > 0$ and $\theta, \eta \in D$. Then we have from Jentzsch's theorem ([20, Theorem V.6.6]) that λ_1 is a simple eigenvalue for $-\mathcal{S}|_D$. Using the standard arguments, like the ones given in the proof of [4, Theorem 2.4], one can show $q_D(t, \theta, \eta)$ is strictly positive when Θ_t is a Brownian motion on S^{d-1} and D is connected or Θ_t is a subordinate Brownian motion on S^{d-1} satisfying the conditions of Example 2.2.

In our analysis the crucial fact is the following lower bound for the eigenvalues λ_j ($j \geq 1$).

Lemma 2.7. *Assume (9) holds true. Then for every $j \geq 1$, we have*

$$(18) \quad \lambda_j \geq [C\sigma(D)]^{-\frac{1}{\beta}} j^{\frac{1}{\beta}}$$

and

$$(19) \quad \|m_j\|_\infty \leq eC [\sigma(D)]^{\frac{1}{2}} \lambda_j^\beta.$$

Proof. We will follow the same idea as the one that the proof of [7, Lemma 2.7] is based on. In particular, since λ_j is ordered increasingly, we have from (17) and (9) that

$$je^{-\lambda_j t} \leq \sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_D q_D(t, \theta, \theta) \sigma(d\theta) \leq C \sigma(D) t^{-\beta}.$$

Taking $t = \lambda_j^{-1}$ we obtain $j \leq C \sigma(D) \lambda_j^\beta$ and (18) follows immediately.

Note that

$$m_j(\theta) = e^{\lambda_j t} Q_t^D m_j(\theta) = e^{\lambda_j t} \int_D q_D(t, \theta, \eta) m_j(\eta) \sigma(d\eta).$$

By the Cauchy-Schwarz inequality,

$$\|m_j\|_\infty \leq e^{\lambda_j t} \sup_\theta \left(\int_D q_D(t, \theta, \eta)^2 \sigma(d\eta) \right)^{1/2} \left(\int_D m_j(\eta)^2 \sigma(d\eta) \right)^{1/2} \leq C [\sigma(D)]^{\frac{1}{2}} e^{\lambda_j t} t^{-\beta}.$$

The proof is completed by setting $t = \lambda_j^{-1}$. □

2.2. Positive self-similar Markov processes. Recall that $R_t = |X_t|$ is a positive (\mathbb{R}_+ -valued) α -self-similar Markov process starting at $|x|$. According to Lamperti [12], up to its first hitting time of 0, R_t may be expressed as a time change of the exponential of an $\mathbb{R} \cup \{-\infty\}$ -valued Lévy process. More formally, there exists an $\mathbb{R} \cup \{-\infty\}$ -valued Lévy process ξ_t starting from 0 and with lifetime ζ , whose law does not depend on $|x|$, such that

$$(20) \quad R_t = |x| \exp(\xi_{A(t)}), \quad 0 \leq t < T_0,$$

where T_0 is the first hitting time of 0 by R defined formally in (5) and $A(t)$ is the positive continuous functional given by $A(t) = \int_0^t R_s^\alpha ds$. The law of ξ is characterized completely by its Lévy-Khintchine exponent

$$(21) \quad \Psi(z) = \log \mathbb{E} [e^{iz\xi_1}] = -q + ibz - \frac{\sigma^2}{2} z^2 + \int_{-\infty}^{+\infty} (e^{izy} - 1 - izy \mathbf{1}_{\{|y|<1\}}) \Pi(dy),$$

where $q \geq 0$, $\sigma \geq 0$, $b \in \mathbb{R}$ and Π is a Lévy measure satisfying the condition $\int_{\mathbb{R}} (1 \wedge |y|^2) \Pi(dy) < \infty$. The lifetime ζ of ξ is an exponential random variable with parameter q , with the convention that $\zeta = \infty$ when $q = 0$. Observe that the process ξ does not depend on the starting point of X . Hence we will denote the law of ξ by \mathbb{P} .

For fixed $\alpha > 0$, we define the exponential functional $I_t(\alpha\xi)$ by (7). Then by a change of variable $s = A(u)$,

$$I_t(\alpha\xi) = \int_0^{A^{-1}(t)} \exp(\alpha\xi_{A(u)}) R_u^{-\alpha} du = |x|^{-\alpha} A^{-1}(t).$$

Hence $A(|x|^\alpha t)$ is the right inverse of the strictly increasing continuous process $I_t(\alpha\xi)$ and we can recover the law of $(R_t, t < T_0)$ from the law of ξ_t for fixed $|x|$ and $\alpha > 0$. In particular, we have

$$(22) \quad (T_0, \mathbb{P}_x) \stackrel{d}{=} (|x|^\alpha I(\alpha\xi), \mathbb{P}).$$

As mentioned in [12], the probabilities $\mathbb{P}_x(T_0 = +\infty)$, $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} = 0)$ and $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} > 0)$ are 0 or 1 independently of x . Moreover, we have

- (i) if $\mathbb{P}_x(T_0 = +\infty) = 1$, then $\zeta = +\infty$, $\limsup_{t \rightarrow \infty} \xi_t = +\infty$, and $\lim_{t \rightarrow \infty} A(t) = +\infty$;
- (ii) if $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} = 0) = 1$, then $\zeta = +\infty$, $\lim_{t \rightarrow \infty} \xi_t = -\infty$, and $\lim_{t \rightarrow T_0-} A(t) = +\infty$;
- (iii) if $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} > 0) = 1$, then ζ is an exponentially distributed random time with parameter $q > 0$. Moreover, $A(T_0-)$ has the same distribution as that of ζ , thus the functional $A(t)$ always jumps from a finite value to $+\infty$, that is, $\mathbb{P}_x(A(T_0-) < +\infty, A(T_0) = +\infty) = 1$.

Let e_λ be an independent exponential random variable with parameter λ . Then

$$\mathbb{E}_{|x|} \left[e^{-\lambda A(t)}, t < T_0 \right] = \mathbb{P}_{|x|} (A(t) < e_\lambda, t < T_0).$$

Note that by the construction above we have that $t < T_0$ is equivalent to $A(t) < \zeta$. Thus

$$(23) \quad \mathbb{E}_{|x|} \left[e^{-\lambda A(t)}, t < T_0 \right] = \mathbb{P}_{|x|} (A(t) < e_\lambda \wedge \zeta) = \mathbb{P} \left(\int_0^{e_\lambda} \exp(\alpha \xi_s) ds > |x|^{-\alpha} t \right),$$

where in the last equality we used the fact that $|x|^\alpha I_t(\alpha \xi)$ is the right inverse of $A(t)$. The equation (23) will give (apart of the representation (17)) another main ingredient of the proof of the main result. In the last step we will need the tail asymptotic behaviour of the exponential function $I_{e_\lambda}(\alpha \xi)$ described below.

2.3. Exponential functional of a killed Lévy process. Let ξ^λ be a Lévy process with Lévy-Khintchine exponent Ψ given by (21) killed by the independent exponential time e_λ with parameter $\lambda > 0$. Thus the resulted process has a lifetime $\zeta' = e_\lambda \wedge \zeta$, an exponential random variable with parameter $\lambda + q$.

We define the Laplace exponent of ξ^λ via

$$(24) \quad \mathbb{E} (\exp(\vartheta \xi_t^\lambda)) = \exp(t\phi(\vartheta)), \quad t \geq 0, \quad \vartheta \in \Xi,$$

where $\Xi = \{\vartheta : \phi(\vartheta) < \infty\}$. By (21), for $\vartheta \in \Xi$ we have

$$(25) \quad \phi(\vartheta) = \Psi(-i\vartheta) = -(q + \lambda) + b\vartheta + \frac{\sigma^2}{2}\vartheta^2 + \int_{-\infty}^{\infty} (e^{\vartheta y} - 1 - \vartheta y \mathbf{1}_{\{|y| < 1\}}) \Pi(dy)$$

for some parameters b, σ and a Lévy measure Π . It is easy to see from Hölder inequality that $\phi(\vartheta)$ is a convex function. From now we assume that ξ satisfies the following conditions.

Assumption 2. ξ^λ is not arithmetic.

Assumption 3. There exists a constant $\vartheta^* > \alpha(1 \vee (2\beta))$ such that $\phi(\vartheta) < \infty$ for $0 < \vartheta < \vartheta^*$ and $\lim_{\vartheta \rightarrow \vartheta^*} \phi(\vartheta) = \infty$, where β is the constant in Assumption 1.

Note that under Assumption 3, we have that for every $\lambda > 0$, there exists a unique $0 < \kappa < \vartheta^*$ such that

$$(26) \quad \phi(\alpha\kappa) = \lambda.$$

Moreover,

$$\mathbb{E}[\xi_1^\lambda e^{\alpha\kappa\xi_1^\lambda}] = \phi'(\alpha\kappa) < +\infty.$$

In the proof we will use the following crucial result giving the tail asymptotics of the distribution of the exponential functional.

Theorem 2.8 ([19, Lemma 4], [18, Theorem 3.1]). *Suppose that the Assumptions 2 and 3 are satisfied. Then, as $t \rightarrow \infty$,*

$$(27) \quad t^\kappa \mathbb{P}(I_{e_\lambda}(\alpha \xi) > t) \sim \frac{1}{\alpha \phi'(\alpha\kappa)} \mathbb{E} [I_{e_\lambda}(\alpha \xi)^{\kappa-1}].$$

Example 2.9 (Linear Brownian motion with drift). Let $\sigma > 0$, $b \in \mathbb{R}$ and $\xi_t = \sigma B_t + bt$, where B_t is a standard linear Brownian motion. Then $\phi(\vartheta) = \frac{\sigma^2 \vartheta^2}{2} + b\vartheta$ and for

$$\kappa = \frac{1}{\alpha \sigma^2} \left(\sqrt{2\sigma^2 \lambda + b^2} - b \right),$$

we have $\mathbb{E} [e^{\alpha\kappa\sigma(B_1+b)}; 1 < e_\lambda] = 1$. By Theorem 2.8,

$$\lim_{t \rightarrow \infty} t^\kappa \mathbb{P}(I_{e_\lambda}(\alpha \xi) > t) = C_\kappa,$$

where

$$C_\kappa = \frac{1}{\alpha\sqrt{2\sigma^2\lambda + b^2}} \mathbb{E} \left\{ \left[\int_0^{e_\lambda} \exp(\alpha\sigma B_s + abs) ds \right]^{\kappa-1} \right\}.$$

By the scaling property of Brownian motion, the random variable

$$\int_0^{e_\lambda} \exp(\alpha\sigma B_s + abs) ds$$

has the same distribution as the integral:

$$\int_0^{\frac{\alpha^2\sigma^2}{4}e_\lambda} \exp\left(2B_s + \frac{4b}{\alpha\sigma^2}s\right) ds.$$

Note that $\frac{\alpha^2\sigma^2}{4}e_\lambda$ is an exponential distributed random variable with parameter $\frac{4}{\alpha^2\sigma^2}\lambda$ independent of B_t . Yor [21] (see also [17, Theorem 4.12]) proved the following the identity in law:

$$(28) \quad I_{e_\lambda}(\alpha\xi) \stackrel{d}{=} \frac{Z_{1,a}}{2\gamma_\kappa},$$

where $a = \kappa + \frac{2b}{\alpha\sigma^2}$, $Z_{1,a}$ is a beta variable with parameters $(1, a)$, and γ_κ is a gamma variable with parameter κ , which is independent of $Z_{1,a}$. Since

$$\mathbb{E}[Z_{1,a}^{\kappa-1}] = \int_0^1 t^{\kappa-1} a(1-t)^{a-1} dt = \frac{\Gamma(\kappa)\Gamma(a+1)}{\Gamma(a+\kappa)}$$

and

$$\mathbb{E}[\gamma_\kappa^{1-\kappa}] = \frac{1}{\Gamma(\kappa)} \int_0^\infty t^{1-\kappa} t^{\kappa-1} e^{-t} dt = \frac{1}{\Gamma(\kappa)},$$

we have

$$\mathbb{E}[(I_{e_\lambda}(\alpha\xi))^{\kappa-1}] = \frac{2^{1-\kappa}\Gamma(a+1)}{\Gamma(\kappa+a)} = \frac{2^{1-\kappa}\Gamma(\kappa + \frac{2b}{\alpha\sigma^2} + 1)}{\Gamma(2\kappa + \frac{2b}{\alpha\sigma^2})}.$$

Therefore,

$$C_\kappa = \frac{4}{\alpha^2\sigma^2 2^\kappa} \frac{\Gamma(\kappa + \frac{2b}{\alpha\sigma^2} + 1)}{\Gamma(2\kappa + \frac{2b}{\alpha\sigma^2} + 1)}.$$

3. MAIN RESULT

Let

$$(29) \quad M(x) = \sum_{j:\lambda_j=\lambda_1} \left(\int_D m_j d\sigma \right) m_j(x/|x|)$$

be a particular eigenfunction corresponding to the eigenvalue λ_1 of the operator \mathcal{S} in D with Dirichlet boundary condition. Moreover, let κ_1 solves $\phi(\alpha\kappa_1) = \lambda_1$, that is, (8) is satisfied.

Recall that τ_C is the exit time for the cone C of the α -self-similar Markov process X_t with a skew-product structure (1). The main result of this paper is the following asymptotics.

Theorem 3.1. *Under the Assumptions 1, 2 and 3, we have,*

$$(30) \quad \mathbb{P}_x(\tau_C > t) \sim \frac{1}{\alpha\phi'(\alpha\kappa_1)} \mathbb{E}[I_{e_{\lambda_1}}(\alpha\xi)^{\kappa_1-1}] M(x) (|x|^{-\alpha}t)^{-\kappa_1},$$

as $t \rightarrow \infty$.

Remark 3.2. $M(x)$ does not depend on the choices of eigenfunctions m_j with $\lambda_j = \lambda_1$. Indeed, if we have another choice m'_j , then there exists an orthogonal matrix (a_{ij}) such that $m'_i = \sum_j a_{ij}m_j$, which is

equivalent to $m_j = \sum_i a_{ij} m'_i$. Thus,

$$\begin{aligned} \sum_i \left(\int_D m'_i d\sigma \right) m'_i &= \sum_i \left(\sum_j \int_D a_{ij} m_j d\sigma \right) m'_i \\ &= \sum_j \left(\int_D m_j d\sigma \right) \sum_i a_{ij} m'_i \\ &= \sum_j \left(\int_D m_j d\sigma \right) m_j. \end{aligned}$$

Example 3.3. Assume X_t is an isotropic α -self-similar diffusion process on \mathbb{R}^d . Then the radial process $R_t = |X_t|$ is a positive α -self-similar diffusion process and Θ_t is a (possibly nonstandard) Brownian motion on S^{d-1} with $a\Delta_{S^{d-1}}$ as its infinitesimal generator for some $a > 0$. Using the Lamperti's relation, we have $\xi_t = \sigma B_t + bt$ for some $\sigma > 0$ and $b \in \mathbb{R}$, where B_t is a standard Brownian motion. Clearly, all the Assumptions 1, 2 and 3 are satisfied. It follows from Example 2.9 that

$$\lim_{t \rightarrow \infty} t^{-\kappa_1} \mathbb{P}_x(\tau_C > t) = \frac{4}{\alpha^2 \sigma^2} \frac{\Gamma(\kappa_1 + \frac{2b}{\alpha \sigma^2} + 1)}{\Gamma(2\kappa_1 + \frac{2b}{\alpha \sigma^2} + 1)} \left(\frac{|x|^2}{2} \right)^{\kappa_1} M(x),$$

where $M(x)$ is defined by (29). In particular, when X_t is a d -dimensional Brownian motion, we have $\alpha = 2$, $\sigma = 1$, $b = \frac{d}{2} - 1$, and $a = \frac{1}{2}$. Thus

$$(31) \quad \lim_{t \rightarrow \infty} t^{-\kappa_1} \mathbb{P}_x(\tau_C > t) = \frac{\Gamma(\kappa_1 + \frac{d}{2})}{\Gamma(2\kappa_1 + \frac{d}{2})} \left(\frac{|x|^2}{2} \right)^{\kappa_1} M(x),$$

where

$$\kappa_1 = \frac{1}{2} \left(\sqrt{2\lambda_1 + \left(\frac{d}{2} - 1 \right)^2} - \left(\frac{d}{2} - 1 \right) \right).$$

This recovers the seminal result of De Blassie [6] (see also [1, Corollary 1]).

Proof of Theorem 3.1. We start the proof from the observation that τ_C is just the first time t that $R_t = 0$ or the angular process $\Theta_{A(t)} \notin D$, that is:

$$(32) \quad \mathbb{P}_x(\tau_C > t) = \mathbb{P}_x(T_0 > t, \tau_D^\Theta > A(t)).$$

By the assumed independence of R_t and Θ_t in (1) we have,

$$(33) \quad \mathbb{P}_x(\tau_C > t) = \int_0^\infty \mathbb{P}_{x/|x|}(\tau_D^\Theta > u) d_u \mathbb{P}_{|x|}(A(t) \leq u, t < T_0).$$

By (17) the exit probability $\mathbb{P}_\theta(\tau_D^\Theta > t)$ can be represented as

$$(34) \quad \mathbb{P}_\theta(\tau_D^\Theta > t) = \int_D q_D(t, \theta, \eta) d\sigma(\eta) = \sum_{j=1}^\infty e^{-\lambda_j t} m_j(\theta) \int_D m_j d\sigma.$$

Thus by (23):

$$\begin{aligned} \mathbb{P}_x(\tau_C > t) &= \sum_{j=1}^\infty m_j(x/|x|) \int_D m_j d\sigma \int_0^\infty e^{-\lambda_j u} d_u \mathbb{P}_{|x|}(A(t) \leq u, t < T_0) \\ (35) \quad &= \sum_{j=1}^\infty m_j(x/|x|) \left(\int_D m_j d\sigma \right) \mathbb{E}_{|x|} \left[e^{-\lambda_j A(t)}, t < T_0 \right] \\ &= \sum_{j=1}^\infty m_j(x/|x|) \left(\int_D m_j d\sigma \right) \mathbb{P}(I_{e_{\lambda_j}}(\alpha \xi) > |x|^{-\alpha} t). \end{aligned}$$

For $j \geq 1$ let κ_j ($j \geq 1$) be the solutions of

$$(36) \quad \phi(\alpha\kappa_j) = \lambda_j.$$

Since $\lambda_j \rightarrow \infty$, we have that $\liminf_{j \rightarrow \infty} \kappa_j \geq \vartheta^*/\alpha$. Fix a κ_0 with the property

$$(37) \quad 1 \vee \kappa_1 \vee (2\beta) < \kappa_0 < \vartheta^*/\alpha.$$

Then there are only a finite number of j 's (for $j \geq 1$), being in the set, say, A , such that $\kappa_1 < \kappa_j \leq \kappa_0$. Applying Theorem 2.8 for $j \in A$ we obtain:

$$\lim_{t \rightarrow \infty} t^{\kappa_1} \mathbb{P} \left(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha} t \right) = 0.$$

Hence

$$(38) \quad \begin{aligned} & \lim_{t \rightarrow \infty} t^{\kappa_1} \sum_{j: \kappa_j \leq \kappa_0} m_j(x/|x|) \left(\int_D m_j d\sigma \right) \mathbb{P} \left(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha} t \right) \\ &= \frac{1}{\alpha\phi'(\alpha\kappa_1)} \mathbb{E} [I_{e_{\lambda_1}}(\alpha\xi)^{\kappa_1-1}] M(x) |x|^{\alpha\kappa_1}. \end{aligned}$$

Now we consider the summation over the $j \in A^c$, that is, for $\kappa_j > \kappa_0$. By the Markov and Hölder inequalities,

$$\begin{aligned} t^{\kappa_0} \mathbb{P} \left(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha} t \right) &\leq |x|^{\alpha\kappa_0} \mathbb{E} \left[\left(\int_0^{e_{\lambda_j}} \exp(\alpha\xi_s) ds \right)^{\kappa_0} \right] \\ &\leq |x|^{\alpha\kappa_0} \mathbb{E} \left[(e_{\lambda_j})^{\kappa_0-1} \int_0^{e_{\lambda_j}} \exp(\alpha\kappa_0 \xi_s) ds \right]. \end{aligned}$$

Using the independence of e_{λ_j} and ξ_s , we have

$$\begin{aligned} t^{\kappa_0} \mathbb{P} \left(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha} t \right) &\leq |x|^{\alpha\kappa_0} \mathbb{E} \left[(e_{\lambda_j})^{\kappa_0} \int_0^{e_{\lambda_j}} e^{s\phi(\alpha\kappa_0)} ds \right] \\ &\leq |x|^{\alpha\kappa_0} \mathbb{E} \left[(e_{\lambda_j})^{\kappa_0} e^{\phi(\alpha\kappa_0)e_{\lambda_j}} \right] \\ &= |x|^{\alpha\kappa_0} \int_0^\infty u^{\kappa_0} e^{\phi(\alpha\kappa_0)u} \lambda_j e^{-\lambda_j u} du \\ &= |x|^{\alpha\kappa_0} \frac{\lambda_j \Gamma(\kappa_0 + 1)}{(\lambda_j - \phi(\alpha\kappa_0))^{\kappa_0+1}} \\ &\leq c |x|^{\alpha\kappa_0} \lambda_j^{-\kappa_0} \end{aligned}$$

for some constant $c > 0$. By (18), (19) and the fact $\kappa_0 > 2\beta$, we have

$$(39) \quad \sum_{j: \kappa_j > \kappa_0} |m_j(x/|x|)| \cdot \left| \int_D m_j d\sigma \right| \mathbb{P} \left(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha} t \right) \leq c |x|^{\alpha\kappa_0} \left(\sum_j j^{-\frac{\kappa_0-\beta}{\beta}} \right) t^{-\kappa_0}.$$

Combining (38) and (39) completes the proof. \square

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